

ON INCOMPLETE LATTICE HOMOMORPHISMS IN SUBSPACES OF GEOMETRIES: “HALF” A PROBLEM OF HARTMANIS FROM 1959

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ABSTRACT. Turing Award winner Juris Hartmanis introduced in 1959 [2] lattices of subspaces of generalized partitions (“partitions of type n ”; “geometries” if $n = 2$). Hartmanis states it is “an unsolved problem whether there are any incomplete lattice homomorphisms in” lattices of subspaces of geometries. (He continues, “[I]f so how can these geometries be characterized.”) We give a positive answer to this question.

1. GENERALIZED PARTITIONS

A *partition of type n* for $n \geq 1$ on a set S (consisting of at least n elements) is a set $\mathfrak{P} \subseteq \mathcal{P}(S)$ such that

- (1) all members of \mathfrak{P} have at least n elements, and
- (2) any n elements of S are contained in exactly one member of \mathfrak{P} .

Partitions of type 1 are the “traditional” partitions.

A partition of type 2 is referred to as a *geometry*, and its elements are called *lines*.

Definition 1.1. *If \mathfrak{G} is a geometry on a set S , a set $T \subseteq S$ is said to be a subspace of S with respect to \mathfrak{G} if it is “closed under lines,” that is, for any distinct $x, y \in T$, for the (unique) element $g \in \mathfrak{G}$ that satisfies $\{x, y\} \subseteq g$ we have $g \subseteq T$.*

We denote the collection of subspaces of S with respect to the geometry \mathfrak{G} by $\text{Sub}(S, \mathfrak{G})$.

If $A \subseteq \text{Sub}(S, \mathfrak{G})$ it is easy to see that $\bigcap A \in \text{Sub}(S, \mathfrak{G})$, therefore $\text{Sub}(S, \mathfrak{G})$ is a complete lattice with respect to set inclusion.

2. INCOMPLETE LATTICE HOMOMORPHISMS

For the terms used in this section, we refer to [1]. Let K, L be complete lattices. If $f : K \rightarrow L$ is order-preserving and $S \subseteq L$ we have

$$f\left(\bigvee_K S\right) \geq f(s) \text{ for all } s \in S,$$

which implies

$$\bigvee_L f(S) \leq f\left(\bigvee_K S\right).$$

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A lattice homomorphism $f : K \rightarrow L$ is said to be *join-incomplete* if there is $S \subseteq K$ non-empty such that $\bigvee_L f(S) < f(\bigvee_K S)$. (Dually, we define *meet-incompleteness*.) We say f is *incomplete* if it is join-incomplete, meet-incomplete, or both.

The next lemmas deal with incomplete lattice homomorphisms in the context of infinite complete distributive lattices.

Lemma 2.1. *Let L be an infinite complete and distributive lattice with bottom element 0 and top element 1. Suppose $P \subseteq L$ is a non-principal prime ideal (i.e., $\bigvee P \notin P$). Then there is a join-incomplete lattice homomorphism $f : L \rightarrow L$ preserving 0 and 1.*

Proof. Let $f : L \rightarrow L$ be 0 on P and 1 on $L \setminus P$. Since P is a prime ideal, $L \setminus P$ is a filter, which implies that f is a lattice homomorphism. It is (join-)incomplete, because $\bigvee P \notin P$ implies $\bigvee f(P) = 0 \neq 1 = f(\bigvee P)$. \square

Of course, there is a dual version of Lemma 2.1 about filters instead of ideals. Next, we show that there is always either a non-principal prime ideal or filter in infinite distributive complete lattices.

Lemma 2.2. *If L is infinite, complete, and distributive, then it contains either a non-principal prime ideal or a non-principal prime filter.*

Proof. Any infinite distributive lattice contains at least a non-principal ideal or a non-principal filter. We may assume that J is a non-principal ideal, so that $j^* = \bigvee J \notin J$. Let $G = \{y \in L : y \geq j^*\}$ be the principal filter generated by j^* . As $J \cap G = \emptyset$ we can use the Prime Ideal Theorem (see [1], Theorem 10.18) and get a prime ideal P such that $J \subseteq P$ and $P \cap G = \emptyset$, which implies $j^* \notin P$.

Next we show that P is not principal: if we had $p^* := \bigvee P \in P$ then $J \subseteq P$ would imply $p^* \geq j^* = \bigvee J$ and $j^* \in P$ because P is a down-set. Therefore P is a non-principal prime ideal. \square

Proposition 2.3. *Let L be an infinite complete and distributive lattice with bottom element 0 and top element 1. Then there is an incomplete lattice homomorphism $f : L \rightarrow L$ respecting 0 and 1.*

Proof. Combine Lemmas 2.1 and 2.2. \square

3. CONSTRUCTION OF AN EXAMPLE

Turing Award winner Juris Hartmanis' problem is on p. 106 of his paper [2]:

So far we have characterized the complete homomorphisms of the lattices of subspaces of geometries. It remains an unsolved problem whether there are any incomplete homomorphisms in these lattices and if so how can these geometries be characterized.

In this section we tackle the first part of Hartmanis' problem.

It asks whether there is a set S and geometry \mathfrak{G} on S and an **incomplete** lattice homomorphism

$$f : \text{Sub}(S, \mathfrak{G}) \rightarrow \text{Sub}(S, \mathfrak{G}).$$

Let $S = \omega$ and set $\mathfrak{G} = \{\{m, n\} : m, n \in \omega \wedge m \neq n\}$.

It is easy to see that $\text{Sub}(\omega, \mathfrak{G}) = \mathcal{P}(\omega)$.

Since $\mathcal{P}(\omega)$ is distributive, Proposition 2.3 shows that it allows an incomplete lattice endomorphism.

In fact, we can give a more constructive way of providing an incomplete lattice homomorphism $f : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$. Let $F \subseteq \mathcal{P}(\omega)$ denote the set of finite subsets of ω and let M denote any maximal ideal containing F . Let f send every member of M to $\emptyset \in \mathcal{P}(\omega)$ and every member of $\mathcal{P}(\omega) \setminus M$ to $\omega \in \mathcal{P}(\omega)$. Then f is an incomplete lattice homomorphism.

What remains open is to have a characterization of the geometries such that the complete lattice of subspaces allows incomplete endomorphisms.

REFERENCES

- [1] B. A. Davey Brian and H. A. Priestley, **Introduction to Lattices and Order** (second edition), Cambridge University Press, 2002.
- [2] Juris Hartmanis, *Lattice Theory of Generalized Partitions*, Canadian Journal of Mathematics **11** (1959), 97-106.

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